§3 Sobolev spaces, elliptic operators

§3.1 Sobolev spaces

Let $\begin{cases} X : \text{compact, oriented, smooth manifold of } \dim_{\mathbb{R}} = n \text{ with Riemannian metric } g. \\ E : \text{vector bundle with metric } h \text{ over } X. \end{cases}$

For $1 \leq p < \infty$, $k \geq 0$, $\phi \in \Gamma(E)$

$$\|\phi\|_{L^p_k} := \left(\int_X \left(|\phi|_h^p + |\nabla_A \phi|_h^p + \cdots + |\nabla_A^k \phi|_h^p \right) d\mathrm{vol}_g \right)^{1/p}$$

Denote by

$L^p_k(X, \Gamma(E))$: the completion by this norm.

(Also $L^p_k(X, \Omega^l(E)) = L^p_k(X, \Gamma(\Lambda^l_X \otimes E))$)

<u>Prop 1</u> (Sobolev embedding)

Suppose $k - \frac{n}{2} > r$, then there is an inclusion map

$$L^2_k(X, \Gamma(E)) \longrightarrow C^r(X, \Gamma(E))$$

r-times continuously differentiable sections

and it is continuous, and compact

<u>Cor</u> if $\varphi \in L^2_k$ for $\forall k$.

then $\varphi \in C^\infty$

<u>Prop 2</u> (Sobolev embedding)

Suppose $k - \frac{n}{p} \geq l - \frac{n}{q}$, $k > l$

Then there is an inclusion map

$$L^p_k(X, \Gamma(E)) \to L^q_l(X, \Gamma(E))$$

and it is continuous.

(i.e. $\|\varphi\|_{L^q_l} \leq {}^\exists C \|\varphi\|_{L^p_k}$ for $\forall \varphi \in L^p_k(X, \Gamma(E))$)

<u>Excercise</u> (Sobolev multiplication)

$n = 4$, Suppose $k \geq l$, $k > 2$, then multiplication map

$$L^2_k \times L^2_l \to L^2_l$$

is defined and it is continuous.

<u>$n=4$</u> if $k > 2$, then we have

$$L^2_k \to C^0 \quad \text{by Prop 1.}$$

Thus $\varphi \in L^2_k$ $(k > 2)$ is continuous. So, we work on L^2_k $(k>2)$-gauge transformations, i.e. L^2_k-bundles $(k>2)$, which are topologically well-defined bundles.

Here, L^2_k-gauge transformations mean gauge transformations of the form $g = g_0 \exp(a)$, where g_0 is a smooth gauge transformation and $a \in L^2_k(X, \Gamma(\mathrm{ad}(P)))$.

Consequently, we work on L^2_{k-1}-connections, L^2_{k-2}-curvatures (cf. Sobolev multiplication) for $k>2$

Denote by

- $\mathcal{A}_{L^2_2}$: the space of L^2_2-connections
- $\mathcal{G}_{L^2_3}$: the space of L^2_3-gauge transformations

<u>Rmk:</u> If X is compact, these spaces are independent of the choices of e.g. metrics and connections.

<u>Facts</u>

- $\mathcal{G}_{L^2_3}$ is a Hilbert Lie group and its Lie algebra is $L^2_3(\Gamma(X, ad(P))$
- action of gauge transformations

$$\mathcal{A}_{L^2_2} \times \mathcal{G}_{L^2_3} \to \mathcal{A}_{L^2_2}$$

is well-defined and it is smooth

- Curvature operator

$$F : \mathcal{A}_{L^2_2} \to L^2_1(X, \Gamma(\Lambda^2 \otimes ad(P))$$

is well-defined and it is smooth.

Define
$$\mathcal{B}_p := \mathcal{A}_{L^2_2} / \mathcal{G}_{L^2_3}$$

Rmk

$\mathcal{B}_p \supset \mathcal{B}_p^*$: all equivalent classes of irreducible connections

defined next lecture.

is Hausdorff.

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§3.2 Elliptic operators.

Let $\begin{cases} X : \text{compact, smooth manifold of } \dim_{\mathbb{R}} = n. \\ E, F \to X, \text{ vector bundles over } X \end{cases}$

$D : \Gamma(E) \to \Gamma(F)$, linear differential operator of order l.

(We extend this to $L^2_{k+l}(X, \Gamma(E)) \to L^2_k(X, \Gamma(E))$)

principal symbol $\sigma_D(\xi, x)$

- consider a neibourhood U of $x \in X$ such that E, F are trivialized.

- write D on U as

$$D = \sum_{|\alpha| \leq l} g_\alpha D^\alpha ,$$

where $g_\alpha \in C^\infty(U, \mathrm{Hom}(E, F))$,

$\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, $D^\alpha = \partial^{|\alpha|} / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$

and $|\alpha| = \alpha_1 + \cdots + \alpha_n$

- Let $\xi = \sum \xi_i dx^i \in T_x^* X$.

and define $\sigma_D(\xi, x)$ by

$$\sum_{|\alpha| = r} g_\alpha(x) \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} .$$

Def

D: elliptic $:\Longleftrightarrow$ $\sigma_D(\xi, x)$ is invertible

for $\forall \xi \neq 0 \in T_x X$

and $\forall x \in X$

Examples

(1) $\Delta := d^* d$, Laplace operator.

(2) Cauchy-Riemann operator $\bar{\partial}$

Prop. (Fredholm property)

D: elliptic operator on a compact manifold

Then $\operatorname{Ker} D$ and $\operatorname{coker} D$ are finite-dimensional.

Prop. (Elliptic estimate)

$D : \Gamma(E) \to \Gamma(F)$: elliptic operator of order ℓ over a compact manifold.

Then for $\forall k \geq 0$, $\exists C_k > 0$ such that for $\forall s \in \Gamma(E)$,

$$\|s\|_{L^2_{k+1}} \leq C_k \left(\|Ds\|_{L^2_k} + \|s\|_{L^2} \right)$$

Elliptic complex

Let $E_i \to X$, vector bundles ($i = 0, \cdots, n$)

and let

$$(*) \quad 0 \to \Gamma(E_0) \xrightarrow{D_0} \Gamma(E_1) \xrightarrow{D_1} \cdots \xrightarrow{D_n} \Gamma(E_n) \to 0$$

be a complex of differential operators of the same order.

Def We call $(*)$ elliptic if

a sequence of principal symbols $\sigma_{D_i}(\xi, x)$

is exact for $\forall \xi \neq 0$, $\forall x \in X$.

Rmk. In this case,

$\sum (D_{even} + D_{odd}^*)$ is an elliptic operator,

where D_{odd}^* is the formal adjoint of D_{odd}.

examples

(1) de Rham complex

(2) Dolbeault complex

§3.3 Linearisation of ASD instanton equation

Let A_0 : ASD

Consider a curve $\{A_t\}$ through A_0 at $t=0$.

Put $\quad \alpha_t := A_t - A_0 \in \Omega^1(\mathrm{ad}(P))$

Then $$\alpha := \frac{d}{dt}\alpha_t \Big|_{t=0}$$

can be thought of as a tangent vector of the curve $\{A_t\}$ at A_0

The curvature of A_t is

$$\begin{aligned} F_{A_t} &= dA_t + A_t \wedge A_t \\ &= F_{A_0} + d_{A_0}\alpha_t + \alpha_t \wedge \alpha_t. \end{aligned}$$

So

$$\frac{d}{dt} F_{A_t}\Big|_{t=0} = d_{A_0}\alpha \qquad \text{as } \alpha_0 = 0$$

Thus the "tangent space" of

$\mathcal{A}_P^{ASD} := \{ A \in \mathcal{A}_P \mid F_A^+ = 0 \}$ at A_0

is given by

$$\text{"} T_{A_0} \mathcal{A}_P^{ASD} \text{"} = \{ \alpha \in \Omega^1(\mathrm{ad}(P)) \mid d_A^+ \alpha = 0 \},$$

where $d_A^+ = P^+ \circ d_A$, $P^+ : \Omega^2(\mathrm{ad}(P)) \to \Omega^+(\mathrm{ad}(P))$

is the projection.

In a similar way, the "tangent space"

"$T_{A_0}(\mathcal{G}_P A_0)$" at A_0 of $\mathcal{G}_P$-orbit of A_0

is given by

$$\text{"} T_{A_0}(\mathcal{G}_P A_0) \text{"} = \{ d_{A_0} a \mid a \in \Omega^0(\mathrm{ad}(P)) \}$$

Atiyah-Hitchin-Singer complex

$$0 \to L^2_3(X, \Lambda^0 \otimes \mathrm{ad}(P)) \xrightarrow{d_A} L^2_2(X, \Lambda^1 \otimes \mathrm{ad}(P))$$

$$\xrightarrow{d_A^+} L^2_1(X, \Lambda^+ \otimes \mathrm{ad}(P)) \to 0 \quad \cdots (*)$$

A : ASD.

Prop.

(*) is an elliptic complex

and the index is given by

$$\left(\frac{1}{2} \dim G \left(e(TX) - \frac{1}{3} p_1(X) \right) - 2 p_1(\mathrm{ad}(P)) \right) . [X]$$

$$\begin{cases} e : \text{Euler class}, \quad p_1 : 1^{st} \text{ Pontryagin class} \\ [X] : \text{fundamental class}. \end{cases}$$

Denote the cohomology of (*) by

$\mathbb{H}^i_A(*)$ $(i = 0, 1, 2)$

The ASD instanton moduli space

$$\mathcal{M}_P := \{A \in \mathcal{A}_{L^2_2} \mid A : \text{ASD}\} \big/ \mathcal{G}_{L^2_3}(P)$$

Fact (elliptic regularity)

Any L^2_2-ASD connection is gauge equivalent to a C^∞ ASD connection by an L^2_3-gauge transformation

(the topology is in fact independent of $l > 2$ for L^2_l.)

Thm (Atiyah-Hitchin-Singer)

If $\mathbb{H}_A^0(*) = \mathbb{H}_A^2(*) = 0$, then

$\mathcal{M}_P$ is a smooth manifold of dim $= -$index.

around $[A]$, and the tangent space at $[A] \in \mathcal{M}_P$ is given by $\mathbb{H}^1_A$.

$G = SU(2)$, $X = S^4$

· $P_1(ad(P)) = -4c_2(E)$

E: associated vector bundle.

· on S^4, $e(TM)[X] = 2$, $\frac{1}{3}P_1(X) = 0$

So if $c_2(E) = 1$, dimension $= 8 - 3 = \underline{5}$.

(in fact $\mathcal{M}_P = B^5$)

So $\mathbb{H}^2_A(*)$ is the obstruction space

to deform $[A] \in \mathcal{M}_P$

Note that $\mathbb{H}^0_A(*)$ is the Lie algebra

of the gauge group.